Linkages Illustrating the Cubic Transformation of Elliptic Functions.

By Colonel R. L. HIPPISLEY, C.B., R.E.

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The connection between poristic polygons and closed linkages, originally pointed out by Prof. Arnold Emch in the 'Annals of Mathematics,' Series 2, vol. 1 (1900), and further developed in the 'Proceedings of the London Mathematical Society,' Series 2, vol. 11 (1911), affords a means of associating linkages with the subject of the transformation of elliptic functions.

Darboux[*] was the first to show systematically how all the variable angles and lines of a three-bar linkage can be expressed in terms of a single elliptic parameter. To him also is due the idea of combining a series of identical three-bar linkages into a closed system; the closure being possible in virtue of the real periodicity of the elliptic functions expressing the variable lines and angles.

A closed linkage of this sort consists of a number of three-bar linkages, all equal in every respect except phase, rotating about the same pair of pivots, and connected together by bars equal in length to the traversing link. Calling the links a, b, c, in left-handed rotation, and d the distance between the pivots, the moving end of the a link of one linkage is connected to the moving end of the c link of the next on the left by a bar of length b, so that the compound linkage, so formed, consists of a series of similar linkages, alternately open and crossed, and in various phases of deformation.

As a general rule, the tangents to the a circle through the moving extremities of the a links (and equally so for the c links) intersect on a fixed conic; but, when the lengths of the a and b links are equal, the conic becomes a circle. If the linkage closes, the polygon formed by the tangents, being circumscribed to one circle and inscribed in another, and being deformable, is a poristic polygon.

The two pivots about which the linkage works are then the foci of a fixed conic which can be inscribed in the polygon formed by the extremities of the a links, and its eccentricity is always d/c.

In order to make the polygon become a triangle, it is necessary that the a, b and c links should all be equal in length; for,

$$c = d\,\frac{1+\mathrm{dn}\frac{2}{3}K}{1-\mathrm{dn}\frac{2}{3}K} \quad\text{and}\quad a = d\,\frac{\mathrm{dn}\frac{2}{3}K}{\mathrm{cn}\frac{2}{3}K\,(1-\mathrm{dn}\frac{2}{3}K)};\text{[†]} \quad\text{and}\quad \frac{\mathrm{dn}\frac{2}{3}K}{\mathrm{cn}\frac{2}{3}K} = 1+\mathrm{dn}\tfrac{2}{3}K.$$

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[*] 'Bulletin des Sciences Mathématiques,' 2nd Series, vol. 3 (1879).

[†] 'Proc. L. M. S.,' Series 2, vol. 11, p. 37.

The only calculation that is then required is that of the distance d, and this is given by

$$d = a\frac{1-\mathrm{dn}\frac{2}{3}\mathrm{K}}{1+\mathrm{dn}\frac{2}{3}\mathrm{K}}.$$

The radius, R, of the circumscribing circle, and the distance, f, of its centre from that of the a circle, are given by

$$\mathrm{R} = a\frac{1+\mathrm{dn}\frac{2}{3}\mathrm{K}}{2\mathrm{cn}\frac{2}{3}\mathrm{K}} \quad \text{and} \quad f = a\frac{1-\mathrm{dn}\frac{2}{3}\mathrm{K}}{2\mathrm{cn}\frac{2}{3}\mathrm{K}}.$$

The circle described by the c link is the inverse of the R circle with reference to the point c, about which the a link revolves, the constant of inversion being $2a^2$. The point Q, where the tangents intersect the R circle, and the point L, the extremity of the c link, are inverse points. This is evident because (fig. 1) $c\mathrm{L}_3 = 2a\cos \mathrm{T}_1c\mathrm{L}_3$ and $c\mathrm{Q}_1 = a/\cos \mathrm{T}_1c\mathrm{L}_3$, so that $c\mathrm{L}_3 \,.\, c\mathrm{Q}_1 = 2a^2$. A positive Peaucellier cell whose arms are a and $a\sqrt{3}$* will, therefore, invert the point L_3 into the point Q_1.

It has been shown by Sir George Greenhill that the centroid of the vertices of a poristic polygon moves in a circle, whose centre is in line with those of the circumscribed and inscribed circles, and that it completes a revolution as the points Q interchange places one step. He has further shown that the angular displacement of the centroid about the centre of its orbit is the sum of the angular displacements of the vertices of the polygon in the circumscribed circle. In the case of the poristic triangle, as its circumcentre is fixed, its orthocentre must also describe a circular orbit; and the same applies to various other points such as the Nägel, Gergonne, Feuerbach and Nine-point centre.

Now the point h, about which c revolves, is the orthocentre of the triangle $\mathrm{T}_1\mathrm{T}_2\mathrm{T}_3$, formed by the extremities of the a links. This is easily seen to be the case, because ch is the vector sum of $c\mathrm{T}_1$, $c\mathrm{T}_2$, and $c\mathrm{T}_3$. As by Weill's theorem the centroid of $\mathrm{T}_1\mathrm{T}_2\mathrm{T}_3$ is fixed, so also is its orthocentre.

We can find the orthocentre of the triangle $\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3$ by applying to the points Q_1, Q_2, Q_3 a linkage of twelve bars exactly similar to the one which generates the points T_1, T_2, T_3. But the points Q_1, Q_2, Q_3 are spaced differently on their circle to the points T_1, T_2, T_3, and their orthocentre H moves. As it describes a circle, it can be connected to a point on the axis by a radial bar.

The radius and centre of this circle can readily be found by placing the triangle $\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3$ in the two positions which are symmetrical with regard to the axis of the figure, and then the orthocentre falls on this axis. In one case, the sides of the triangle are inclined to the axis at an angle am $\frac{1}{3}$K, and, in the other, at an angle $\frac{1}{2}\pi - \text{am}\,\frac{2}{3}\text{K}$; whence we have

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* More generally, in the case of the triangle, $\frac{1}{2}(\mathrm{R}-a+d+f)$ and $\frac{1}{2}(\mathrm{R}+a-d+f)$, the difference of whose square is $2a^2$.

$$a_2 h_1 = 2\,\text{R}\,\text{cn}^2\tfrac{2}{3}\text{K}, \qquad h_2 a_1 = 2\,\text{R}\,\text{sn}^2\tfrac{1}{3}\text{K},$$

and, remembering that $\text{cn}\frac{2}{3}\text{K}\,(1+\text{dn}\frac{2}{3}\text{K}) = \text{dn}\frac{2}{3}\text{K}$, we obtain without difficulty

$$c\text{O}' = \text{R}\,\frac{1-\text{dn}\frac{2}{3}\text{K}}{1+\text{dn}\frac{2}{3}\text{K}} = f, \qquad \text{O}'\text{H} = \text{R}\,\frac{(1-\text{dn}\frac{2}{3}\text{K})^2}{(1+\text{dn}\frac{2}{3}\text{K})^2} = \frac{f^2}{\text{R}} = h, \text{ say},$$

so that

$$f^2 = h\text{R}.$$

The arm h, therefore, integrates the angular motion of the arms CQ_1, CQ_2 and CQ_3. This can be easily proved analytically, for, denoting the radial angles of the points T_1, T_2 and T_3, by α_1, α_2 and α_3, those of Q_1, Q_2, and Q_3 by $2\phi_1$, $2\phi_2$, and $2\phi_3$, and that of the arm h by 2Φ, we have

$$\alpha_1 = \phi_1+\phi_2, \qquad \alpha_2 = \phi_2+\phi_3, \qquad \alpha_3 = \phi_3+\phi_1+\pi,$$

therefore

$$a\left(e^{\iota(\phi_1+\phi_2)} + e^{\iota(\phi_2+\phi_3)} - e^{\iota(\phi_3+\phi_1)}\right) = d. \tag{1}$$

We also have

$$he^{2\iota\Phi} = \text{R}\,(e^{2\iota\phi_1} + e^{2\iota\phi_2} + e^{2\iota\phi_3}) - 2f. \tag{2}$$

Multiplying (1) by $e^{-\iota(\phi_1+\phi_2+\phi_3)}$ and changing ι into $-\iota$ in the result, we get

$$a\left(e^{\iota\phi_1} - e^{\iota\phi_2} + e^{\iota\phi_3}\right) = de^{\iota(\phi_1+\phi_2+\phi_3)}$$

and squaring

$$a^2\left\{e^{2\iota\phi_1} + e^{2\iota\phi_2} + e^{2\iota\phi_3} - 2e^{\iota(\phi_1+\phi_2)} - 2e^{\iota(\phi_2+\phi_3)} + 2e^{\iota(\phi_3+\phi_1)}\right\} = d^2 e^{2\iota(\phi_1+\phi_2+\phi_3)}$$

which is

$$\frac{a^2(he^{2\iota\Phi}+2f)}{\text{R}} - 2ad = d^2 e^{2\iota(\phi_1+\phi_2+\phi_3)},$$

or

$$he^{2\iota\Phi} + 2f = \frac{2\,\text{R}d}{a} + \frac{\text{R}d^2}{a^2}\,e^{2\iota(\phi_1+\phi_2+\phi_3)}.$$

But

$$\frac{\text{R}d}{a} = f \qquad \text{and} \qquad h = \frac{f^2}{\text{R}} = \frac{\text{R}d^2}{a^2},$$

therefore

$$\Phi = \phi_1+\phi_2+\phi_3.$$

Now ϕ is the amplitude of the elliptic function by which the position of the point Q is expressed,* therefore Φ is the sum of the amplitudes.

The formula of the 'Fundamenta Nova,' § 23 (19),

$$\frac{1}{\text{M}}\,\text{dn}\left(\frac{u}{\text{M}}, \lambda\right) = \text{dn}\,u + \sum_{r=1}^{\frac{1}{2}(n-1)} \text{dn}\,(u \pm 2r\omega),$$

integrates immediately, as pointed out by Sir George Greenhill, into

$$\text{am}\left(\frac{u}{\text{M}}, \lambda\right) = \text{am}\,u + \Sigma\,\text{am}\,(u \pm 2r\omega),$$

but $\phi_r = \text{am}\,(u+2r\omega)$, hence Φ is the amplitude of the transformed function and the movement of the point H gives for the transformed function that which the movement of Q gives for the original function.

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* Greenhill's 'Elliptic Functions,' pp. 4, 124.

The linkage, then, consists of three parts. Firstly, the closed linkage $T_1L_1T_2L_2T_3L_3$, formed by the series of equal three-bar linkages cT_1L_1h, cT_2L_1h, cT_2L_2h, etc., alternately open and crossed, which gives the points of contact T_1, T_2, T_3, of the poristic triangle with its inscribed circle, and incidentally the points L_1, L_2, L_3, which are those of the same poristic triangle, differing in phase from Q_1, Q_2, Q_3, by $\frac{1}{3}$K, but having different centering; secondly, three Peaucellier cells (shown by broken lines in fig. 1), which indicate the vertices of the triangle; and, thirdly, another closed linkage pivoted to the points

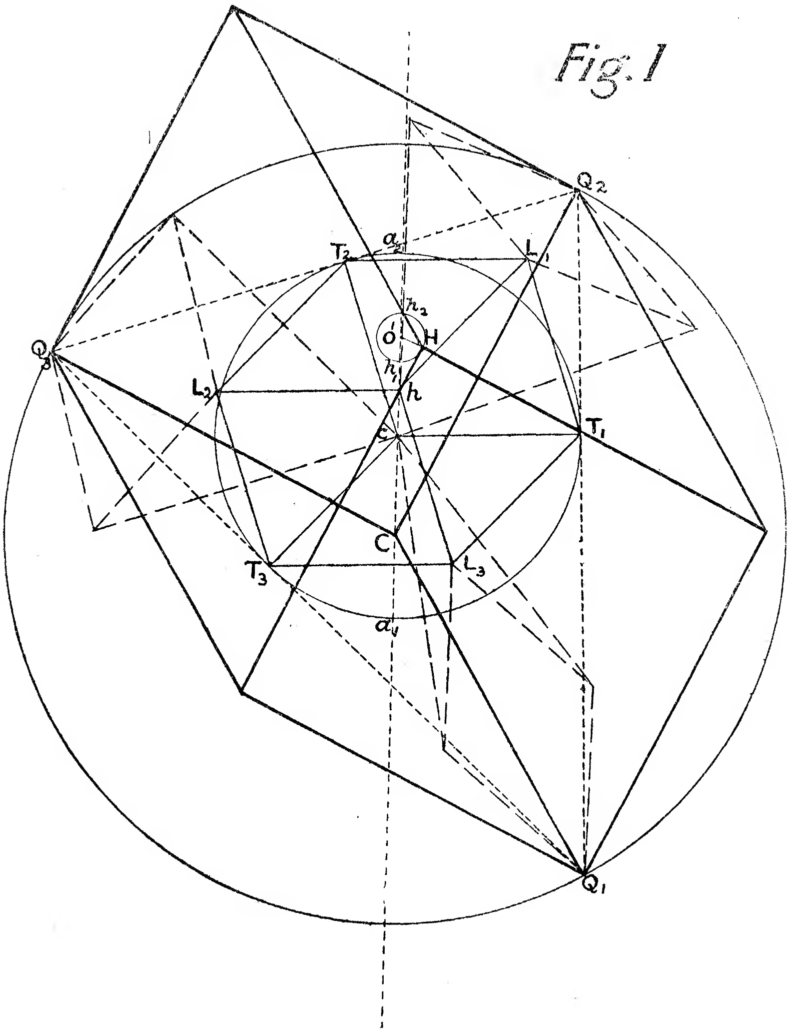
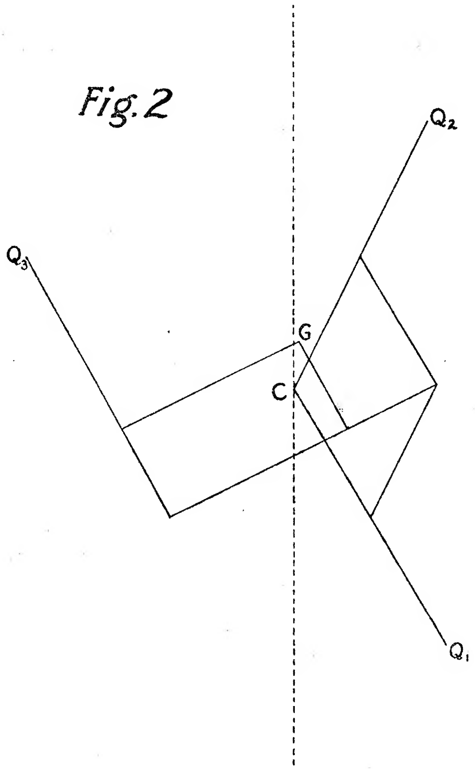


Fig. 1

Q_1, Q_2, Q_3, similar to the first, but enlarged to suit the circle R, which finds the orthocentre H. The point C of this linkage is stationary, and may be fixed by a pivot to give stiffness to the mechanism, but this is not essential.

If preferred, this third part can be replaced by an attachment which will point out the centroid of $Q_1Q_2Q_3$. This consists of two pantagraphs Q_1CQ_2 and Q_3G (fig. 2), one set to the ratio $\frac{1}{2}$, and the other to the ratio $\frac{1}{3}$. A link (not shown) may be inserted between C and Q_3 for the sake of stiffness.

Fig. 2



Added May 30, 1916.—The principle upon which the linkage for the cubic transformation is based is clearly applicable to one for any order; but, as n increases, the number of links required increases so rapidly that it becomes necessary to economise in their use.

Accordingly the Peaucellier cells should be replaced by crossed parallelograms, which, suitably proportioned, will perform the same function, thus saving four links for each Q point. These crossed parallelograms hNMC (fig. 3) will have sides $c+R$ and $f+d$, and fixed centres at h and C; then a point on the link CM, at a distance R from C, will indicate Q.

Fig. 3 shows the Peaucellier cell and the crossed parallelogram for one Q point. The line EF envelops a conic whose foci are at h and C. The radii hL and CQ of the circles described by L and Q intersect at the point s where EF touches its envelope, and are therefore inclined to the axis hC at the

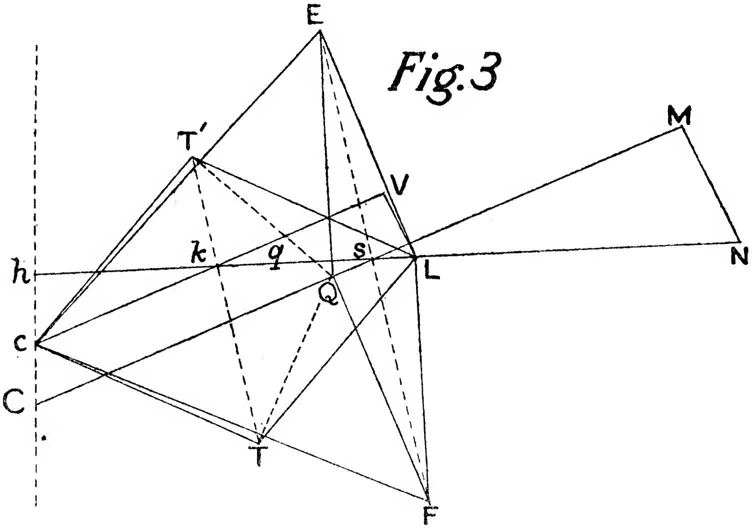


Fig. 3

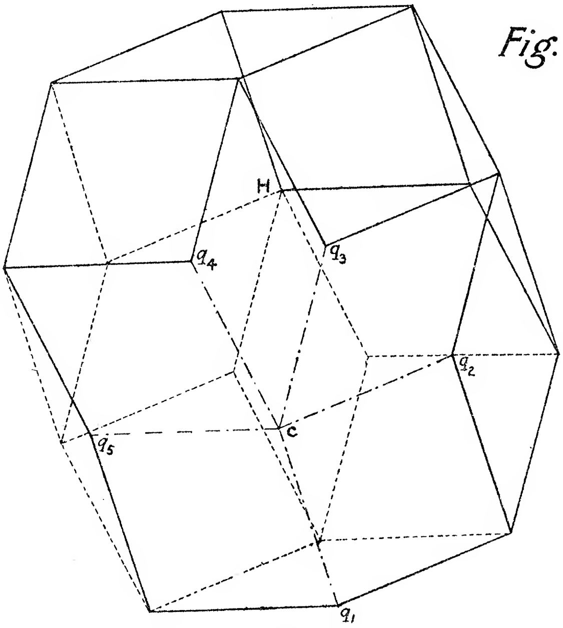


Fig. 4

focal angles of the point s. The major axis of this conic is $h\text{L}+\text{CQ}$. The crossed parallelogram hNMC, where hN and MC are both of length $c+\text{R}$, will produce this conic at the point s, and therefore also the point Q.

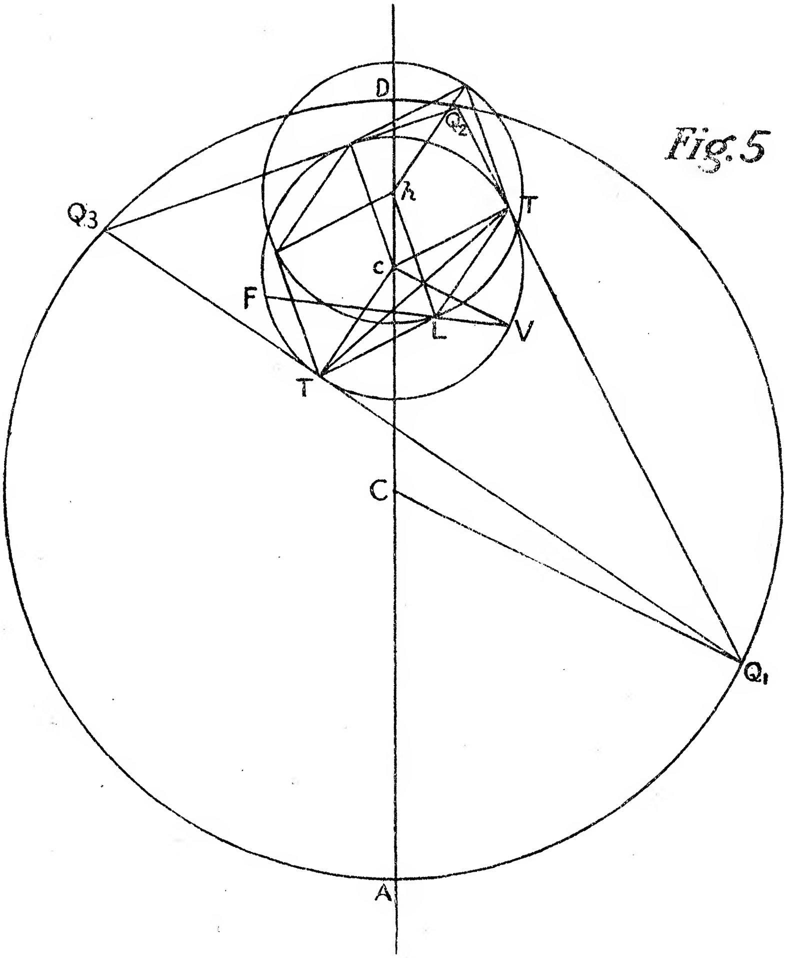


Fig. 5

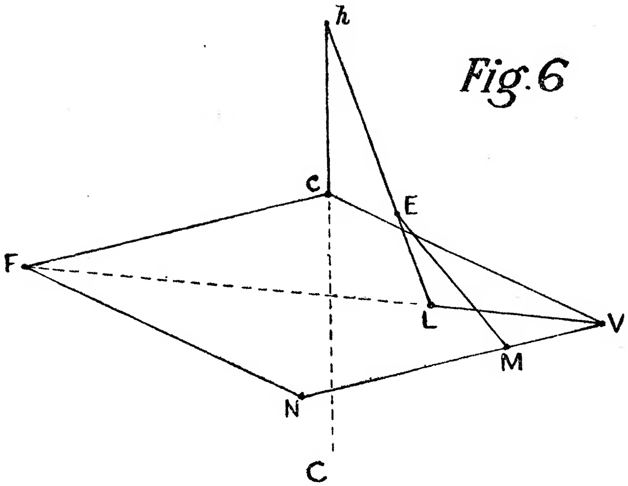


Fig. 6

Though it is satisfactory to have ocular demonstration of the points Q, it is not necessary to display them from the point of view of the Transformation Theory. A polygon similar to the Q polygon will serve the purpose equally well. It is mechanically preferable to reduce the size of the crossed parallelogram, so as to keep all the links of proportionate size. For economy in links it is convenient to centre the reduced polygon at the point c, which simply means shortening the links in the proportion of hc to hC, or as $d : d+f$. The long link then becomes of length c, because $(\mathrm{R}+c)\, d/(d+f) = c$,* and the crossed parallelogram is $h$LV$c$.

Now h and c are the foci of the envelope of TT′, which is known to touch its envelope where hL cuts it. Its focal angles are also known to be the inclinations of hL and CQ to the axis; hL and cV therefore intersect on TT′.

The linkage for finding the vector sum of the radii CQ or cq, in the case of the pentagon, is shown in fig. 4, which hardly needs explanation. The points $q_1, q_2, q_3, \ldots$, correspond to the various points q of fig. 3, and the vector sum of cq_1, cq_2, $cq_3 \ldots$ is cH. H therefore describes a circle with its centre on the axis Ch, and its radial angle is the double amplitude of the transformed function.

Only the links shown in firm lines are necessary, but the remainder are added to complete the figure and to indicate that more than one way of approach to the vector sum is possible.

In the case of the poristic triangle, the line VL (fig. 5) passes through the Feuerbach point, F, of the triangle $\mathrm{Q_1Q_2Q_3}$, which is on the inscribed circle $\mathrm{T_1T_2T_3}$. The point F is therefore where VL again meets the circle $\mathrm{T_1T_2T_3}$, and will be at the corner of a rhombus formed on FcV. If, therefore (fig. 6), we keep the angle LVN equal to cVL by one of A. B. Kempe's reversor linkages, and complete the rhombus, we shall determine F.

Now hL is perpendicular to $\mathrm{Q_2Q_3}$. The angle chL is therefore $\phi_2+\phi_3-\pi$, and cVN is $2(\phi_2+\phi_3)-2\pi$; while $\mathrm{C}c\mathrm{V} = \mathrm{ACQ_1} = 2\phi_1$. But $\mathrm{F}ch = \pi - \mathrm{F}c\mathrm{C} = \pi - \mathrm{F}c\mathrm{V} + \mathrm{C}c\mathrm{V} = c\mathrm{VN} + \mathrm{C}c\mathrm{V}$; therefore $\mathrm{F}ch = 2(\phi_1+\phi_2+\phi_3)-2\pi$, and F$ch$ is the double amplitude of the transformed function.

The cubic transformation linkage, then, in its simplest form is that shown in fig. 6, the inclination of cV to cC being the double amplitude of (u, k) and the inclination of cF to ch being that of $(u/\mathrm{M}, \lambda)$.

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* 'Proc. L. M. S.,' *loc. cit.*